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P^2-REDUCING AND TOROIDAL DEHN FILLINGS

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Abstract. We study the situation where we have two exceptional Dehn fillings on a given hyperbolic 3-manifold. We consider two cases that one filling creates a projective plane, and the other creates an essential torus or a Klein bottle, and give the best possible upper bound on the distance between two fillings for each case.

1. Introduction

Let M be a compact, connected, orientable 3-manifold with a torus boundary component $\partial_0 M$. A *slope* on $\partial_0 M$ is the isotopy class of an unoriented essential simple loop. For a slope r, the manifold obtained from M by *r-Dehn filling* is $M(r) = M \cup V_r$, where V_r is a solid torus glued to M along $\partial_0 M$ in such a way that r bounds a meridian disk in V_r. If r and s are two slopes on $\partial_0 M$, then $\Delta(r, s)$ denotes their minimal geometric intersection number.

We say that a 3-manifold M is *hyperbolic* if M with its boundary tori removed admits a complete hyperbolic structure of finite volume with totally geodesic boundary. If M has non-empty boundary, then Thurston's geometrization theorem for Haken manifolds [19] says that M is hyperbolic if and only if M contains no essential sphere, disk, torus or annulus.

We are interested in obtaining restrictions on when a Dehn filling on a hyperbolic 3-manifold fails to be hyperbolic. Such a filling is said to be *exceptional*. It is well known that if M is hyperbolic, then there are only finitely many exceptional Dehn fillings on $\partial_0 M$ [19], and there are a large amount of investigations on exceptional Dehn fillings (see [8]).

In this paper, we deal with three specific exceptional Dehn fillings. A 3-manifold is *P^2-reducible* if it contains a projective plane, and *P^2-irreducible* otherwise. If a closed 3-manifold is P^2-reducible, then it is either the real 3-dimensional projective space P^3 or a reducible manifold with a P^3-summand. A 3-manifold is *toroidal* if it contains an essential torus. Clearly, if $M(r)$ is P^2-reducible or toroidal, then such a filling is exceptional. Finally a hyperbolic 3-manifold contains no Klein bottle, and if $M(r)$ contains a Klein bottle, then the filling is also exceptional (see [16]).

We consider two situations. That is, one filling yields a P^2-reducible manifold, and the other gives either a toroidal manifold or a manifold containing a Klein bottle. For each case, we can find the best possible upper bound on the distance between two exceptional fillings.

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Theorem 1.1. *Let M be a hyperbolic 3-manifold with a torus boundary component $\partial_0 M$. Let α and β be two slopes on $\partial_0 M$ such that $M(\alpha)$ is P^2-reducible and $M(\beta)$ is toroidal. Then either*

(1) $\Delta(\alpha,\beta)\le 2$*; or*
(2) $\Delta(\alpha,\beta)=3$ *and $M(\beta)$ contains an essential torus which intersects the core of the attached solid torus in two points.*

Theorem 1.2. *Let M be as in Theorem 1.1. Let α and β be two slopes on $\partial_0 M$ such that $M(\alpha)$ is P^2-reducible and $M(\gamma)$ contains a Klein bottle. Then either*

(1) $\Delta(\alpha,\gamma)\le 2$*; or*
(2) $\Delta(\alpha,\gamma)=3$ *and $M(\gamma)$ contains a Klein bottle which intersects the core of the attached solid torus in a single point.*

The examples showing that these estimates are sharp are given in the final section.

In [6], Gordon gave an upper bound 5 for the distance between a toroidal filling and a lens space filling on a hyperbolic 3-manifold with torus boundary. Our Theorem 1.1 gives a partial improvement of this result in case where a lens space is the real projective 3-space P^3.

Corollary 1.3. *Let M be a hyperbolic 3-manifold with torus boundary ∂M, and let α and β be slopes on ∂M such that $M(\alpha)$ is the lens space $L(2,1)$ $(=P^3)$ and $M(\beta)$ is toroidal. Then $\Delta(\alpha,\beta)\le 3$.*

The authors would like to thank Cameron Gordon for suggesting the problem.

2. Preliminaries

In the remainder of this paper, we assume that M is a hyperbolic 3-manifold with a torus boundary component $\partial_0 M$, and that α,β,γ are slopes on $\partial_0 M$ such that $M(\alpha)$ is P^2-reducible, $M(\beta)$ is toroidal, and $M(\gamma)$ contains a Klein bottle.

Assume that $\Delta(\alpha,\beta),\Delta(\alpha,\gamma)\ge 3$. Suppose that $M(\alpha)$, $M(\beta)$ and $M(\gamma)$ contain a projective plane $\widehat{P}$, an essential torus $\widehat{T}$ and a Klein bottle $\widehat{K}$ respectively. Then we may assume that $\widehat{P}$ meets the attached solid torus V_α in a finite collection of meridian disks, so that $P=\widehat{P}\cap M$ is a punctured projective plane properly embedded in M, each of whose boundary components has slope α. Furthermore, we can assume that $\widehat{P}$ is chosen so that the number of boundary components p is minimal among all projective planes in $M(\alpha)$. Similarly, $\widehat{T}$ and $\widehat{K}$ give rise to the surfaces T and K respectively, and the numbers of boundary components t and k of T and K are assumed to be minimal. Recall that $M(\alpha)$ is either the real projective 3-space P^3, or a reducible manifold with a P^3-summand. Therefore we may assume that $M(\beta)$ and $M(\gamma)$ are irreducible by [2, 10].

Lemma 2.1. $p\ge 2$.

Proof. If $p=0$, then M contains a projective plane, which is impossible since M is hyperbolic. If $p=1$, then M contains a Möbius band, which is also impossible. □

Lemma 2.2. *P is incompressible and boundary-incompressibe in M.*

Proof. Assume P is compressible in M. Let D be a compressing disk for P. Note that ∂D is orientation-preserving on P (and hence $\widehat{P}$). Hence ∂D bounds a disk D' on $\widehat{P}$. Since $\mathrm{Int}D'$ meets V_α, we can create a new projective plane by replacing D' with D, which meets V_α fewer than $\widehat{P}$. This contradicts the minimality of $\widehat{P}$. Therefore P is incompressible in M.

Next, assume that P is boundary-compressible. Then P would be compressible, or the core of V_α can be isotoped into $\widehat{P}$ as an orientation-reversing loop. But, the latter case implies that M is boundary-reducible. □

Since t is minimal, it is clear that T is incompressible and boundary-incompressible in M. We have $t \geq 1$, and $k \geq 1$, since M cannot contain a Klein bottle.

Lemma 2.3. *K is incompressible and boundary-incompressible in M.*

Proof. Suppose that K is compressible in M. Let D be a disk in M such that $D \cap K = \partial D$ and ∂D does not bound a disk on K. Note that ∂D is orientation-preserving on K.

If ∂D is non-separating on $\widehat{K}$, then we get a non-separating 2-sphere in $M(\gamma)$ by compressing $\widehat{K}$ along D. This is clearly a contradiction. If ∂D bound a disk on $\widehat{K}$, then we replace the disk with D, and get a new Klein bottle in $M(\gamma)$ with fewer intersections with V_γ than $\widehat{K}$. This contradicts the choice of $\widehat{K}$.

Thus ∂D is essential and separating on $\widehat{K}$. Compressing $\widehat{K}$ along D gives two disjoint projective planes in $M(\gamma)$. Since $M(\gamma)$ is irreducible, this is also impossible. Thus we have shown that K is incompressible.

Next, let E be a disk in M such that $E \cap K = \partial E \cap K$, $\partial E = a \cup b$, where $a \subset K$ is an essential (i.e. not boundary-parallel) arc in K and $b \subset \partial_0 M$. If a joins distinct components of ∂K, then a compressing disk for K is obtained from two parallel copies of E and the disk obtained by removing a neighborhood of b from the annulus in $\partial_0 M$ cobounded by those components of ∂K meeting a. Hence ∂a is contained in the same component $\partial_1 K$, say, of ∂K. If $k \geq 2$, then b bounds a disk D' in $\partial_0 M$ together with a subarc of $\partial_1 K$. Then $E \cup D'$ gives a compressing disk for K in M. Therefore $k = 1$. Then we can move the core of V_γ onto an orientation-reversing loop in $\widehat{K}$ by using E. This implies that M contains a properly embedded Möbius band, which contradicts the fact that M is hyperbolic. □

From the arc components of $P \cap T$, $P \cap K$, we can construct two pairs of graphs (G_P^T, G_T) and (G_P^K, G_K) in the usual way (see [3, 7, 11]). We number the components of ∂T as $1, 2, \ldots, t$ in the order in which they appear on $\partial_0 M$. Similarly number the components of ∂K. But the components of ∂P are numbered from -1 to $p-2$ unusually. For simplifying the notations, we use the symbol G_P for both G_P^T and G_P^K, and G_{TK} for G_T or G_K. For a graph G, the *reduced graph* $\overline{G}$ of G is defined to be the graph obtained from G by amalgamating each family of parallel edges into a single edge.

Lemma 2.4. *Neither G_P nor G_{TK} has trivial loops.*

Proof. This follows from the fact that P, T and K are boundary-incompressible. □

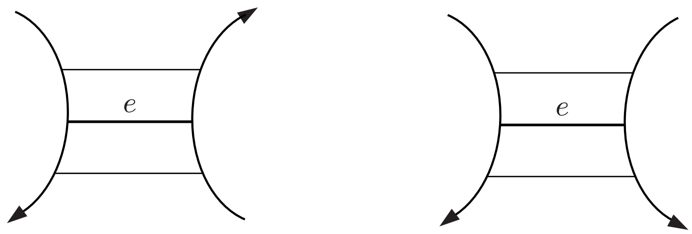


FIGURE 1. A sign of an edge

We may assume that no circle component of $P \cap T$ or $P \cap K$ bounds a disk in P, T or K, because of the incompressibilities of these surfaces.

Although P and K are non-orientable, we can establish a parity rule, which plays a crucial role in this paper. In fact, this is a natural generalization of the usual parity rule [3].

First, orient all components of ∂P so that they are mutually homologous on $\partial_0 M$. Similarly for ∂T and ∂K. Let e be an edge in G_P. Since e is an arc properly embedded in P, a regular neighborhood D of e in P is a disk in P. Then $\partial D = a \cup b \cup c \cup d$, where a and c are arcs in ∂P with induced orientations from ∂P. On D, if a and c have opposite directions, then e is called *positive*, otherwise *negative*. See Figure 1. Similarly, define the sign of edges in G_{TK}. Then we have the following rule.

Lemma 2.5 (Parity rule)**.** *An edge e is positive (or negative) in G_P if and only if e is negative (positive resp.) in G_{TK}.*

Proof. This follows from the fact that M is orientable and $\partial_0 M$ is a torus. □

Remark that G_T can have positive loops, but no negative loops. G_P and G_K can have positive and negative loops. But the key point is;

Lemma 2.6. (1) *At most one vertex can be a base of negative loops in G_P.*
(2) *At most two vertices can be bases of negative loops in G_K.*

Proof. Let e be a negative loop based at a vertex x. Then $N(x \cup e)$ is a Möbius band. Since $\widehat{P}$ and $\widehat{K}$ can contain at most one Möbius band and at most two disjoint Möbius bands respectively, the conclusions follow. □

Now, we have some basic properties of the graphs. An edge is called an *x-edge* if it has label x at its endpoint, and a *level edge* if both endpoints have the same label. For example, G_{KT} can have a positive level edge, which corresponds to a negative loop in G_P by the parity rule. Thus at most one label of G_{TK} can be a label of positive level edges by Lemma 2.6(1). Therefore, we adopt the convention that 0 is the label of positive level edges in G_{TK}.

Let $G = G_P$ or G_{TK}. A cycle in G is a *Scharlemann cycle* if it bounds a disk face, and the edges in the cycle are all positive and have the same label pair $\{i, i+1\}$ at their two endpoints, called the *label pair* of the Scharlemann cycle. (In this case,

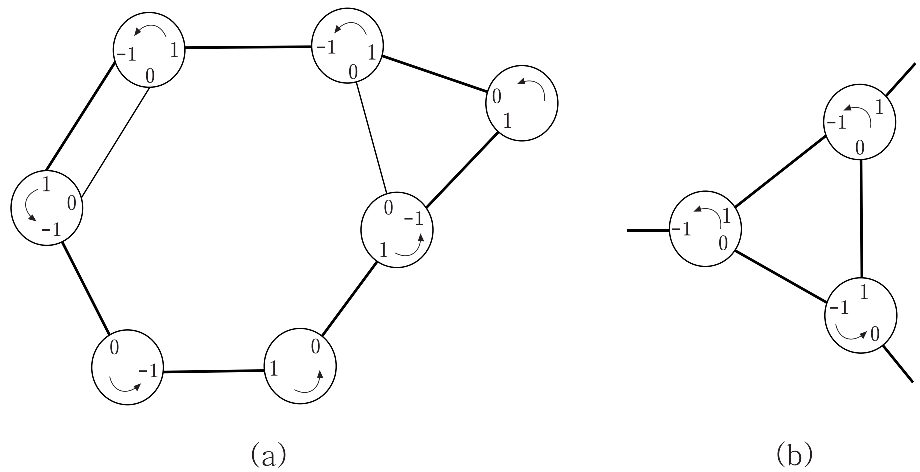


FIGURE 2. A generalized Scharlemann cycle

the label set of G must have at least two elements.) In particular, a Scharlemann cycle of length two is called an *S-cycle* in short.

When $p \geq 3$, a *generalized Scharlemann cycle* in G_{TK} is defined to be either a Scharlemann cycle or a cycle of positive edges whose labels are in the set $\{-1, 0, 1\}$ at both ends and which bounds a disk face in $G_{TK} - \{\text{level edges}\}$. In addition, if $p = 3$, then the label 0 must appear around each vertex in the disk face bounded by the generalized Scharlemann cycle in $G_{TK} - \{\text{level edges}\}$. For example, Figure 2(b) is not a generalized Scharlemann cycle. Refer to [4] for more details.

A generalized Scharlemann cycle of length two is called a *generalized S-cycle*. Then a generalized S-cycle, not an S-cycle, in G_{TK} is a triple $\{e_{-1}, e_0, e_1\}$ of mutually parallel positive edges where e_{-1} and e_1 have the same label pair $\{-1, 1\}$, and e_0 is a level edge with label 0. A generalized S-cycle, not an S-cycle, can be defined in G_P (precisely, G_P^K), but then the label set is $\{i-1, i, i+1\}$ for some i.

Lemma 2.7. *G_{TK} contains no generalized Scharlemann cycles.*

Proof. This is [4, Theorem 1.1]. □

Lemma 2.8. (1) *If p is even, G_{TK} has at most $p/2$ mutually parallel positive edges. Furthermore, if there is a family of $p/2$ mutually parallel positive edges, then either the first, say, edge of the family is level, or the set of labels of one end of the family is disjoint from that of the other end.*

(2) *If p is odd, G_{TK} has at most $(p+1)/2$ mutually parallel positive edges. Furthermore, if there is a family of $(p+1)/2$ mutually parallel positive edges, then the first, say, edge of the family is level.*

Proof. These follow from Lemmas 2.6(1) and 2.7. □

Lemma 2.9. *Let $p \geq 2$. Then G_{TK} cannot contain p mutually parallel edges.*

Proof. Let $A_1, A_2, \dots, A_p$ be a family of p mutually parallel edges in G_{TK} labelled successively. By Lemma 2.8, all A_i's are negative, and then they make orientation-preserving cycles in G_P. Note that an orientation-preserving loop in a projective plane is contractible. Therefore, we can choose an innermost cycle among them. Then the construction in [9, Section 5] implies that M is cabled, a contradiction. $\square$

3. Main argument

In this section, we use $|G_{TK}|$ to denote the number of vertices of G_{TK}.

For a label x of G_{TK}, let Γ_x be the subgraph of G_{TK} consisting of all the vertices and positive x-edges of G_{TK}. A disk face D of Γ_x is called an *x-face* of G_{TK}.

Let G_P^+ denote the subgraph of G_P consisting of all the vertices of G_P and the positive edges of G_P. Note that G_P^+ has a disk support in $\widehat{P}$, that is, there is a disk in $\widehat{P}$ which contains G_P^+ in its interior, since any orientation-preserving loop in a projective plane is contractible.

Let Λ be a subgraph of G_P^+ with a disk support D. A vertex of Λ is a *boundary vertex* if there is an arc connecting it to ∂D whose interior is disjoint from Λ, and an *interior vertex* otherwise.

A *generalized web* Λ_P is a connected subgraph of G_P^+ satisfying that

(i) at most one boundary vertex y of Λ_P is a cut vertex of G_P^+;
(ii) each vertex of Λ_P, except y if it exists, has degree at least $(\Delta-1)|G_{TK}|$ in Λ_P;
(iii) there is a disk D in $\widehat{P}$ such that $D \cap G_P^+ = \Lambda_P$.

The vertex y as in (i), if it exists, is called an *exceptional vertex* of Λ_P.

Proposition 3.1. *One of the followings holds;*

(1) *G_{TK} contains an x-face for some $x \neq 0$;*
(2) *G_P contains a generalized web.*

Proof. We distinguish two cases.

Case 1: Suppose that there is a vertex $x(\neq 0)$ of G_P such that more than $|G_{TK}|$ negative edges are incident to x. Remark that such negative edges are not loops because $x \neq 0$. This implies that there exist more than $|G_{TK}|$ positive x-edges in G_{TK} by the parity rule. Thus Γ_x has a larger number of edges than that of vertices. An Euler characteristic calculation gives that Γ_x contains a disk face.

Case 2: As the negation of Case 1, suppose that each vertex $x(\neq 0)$ of G_P has at least $(\Delta-1)|G_{TK}|$ positive edge endpoints.

Let Λ be an extremal component of G_P^+. That is, Λ is a component of G_P^+ having a disk support D such that $D \cap G_P^+ = \Lambda$.

First, assume that Λ is a single vertex. Then it must be vertex 0. Thus only negative edges are incident to vertex 0 in G_P. If there is no negative loop at vertex 0, then Λ can be the only extremal component of G_P^+. But then, G_{TK} contains a 0-face E as in Case 1, and G_{TK} has no positive level edges. Then E contains a Scharlemann cycle by [14, Proposition 5.1], which is impossible by Lemma 2.7. If a

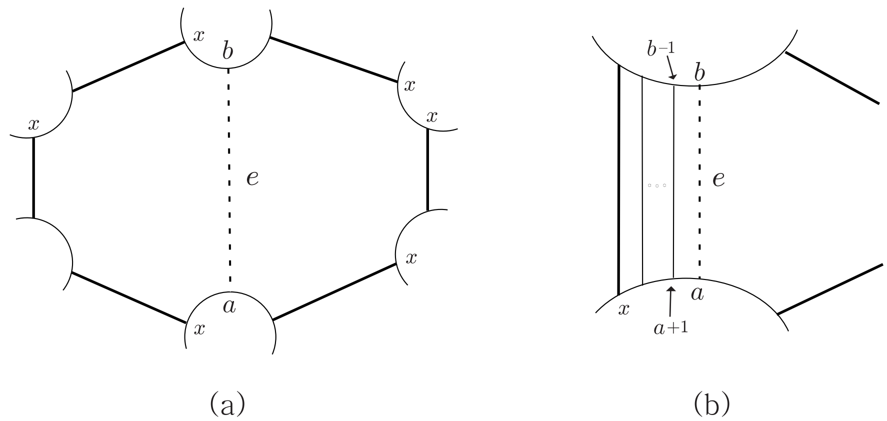


FIGURE 3. Split along a diagonal edge

negative loop is incident there, we can choose another extremal component of G_P^+, which has more than one vertex. Thus we can assume that Λ is not a single vertex.

Choose a block Λ_P of Λ with at most one cut vertex. Then Λ_P is clearly a generalized web. □

4. x-FACE OF G_{TK}

In this section, we treat the case (1) of Proposition 3.1.

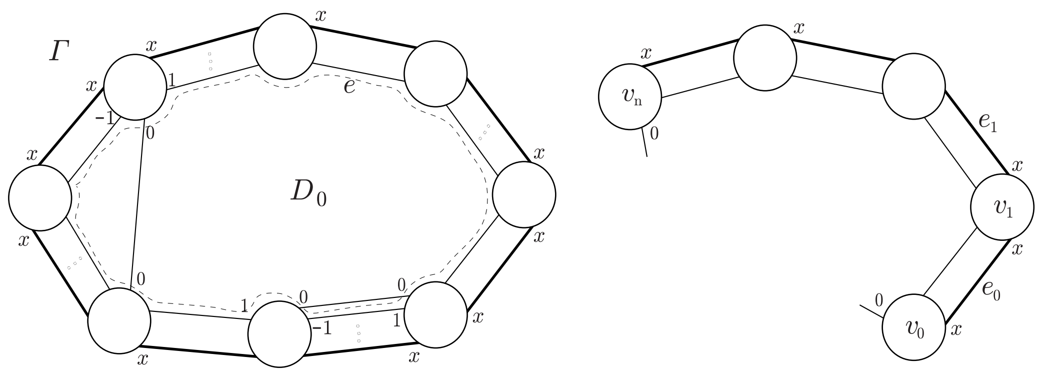
Theorem 4.1. *Let $p \geq 3$. If G_{TK} contains a non-zero x-face D, then it contains a generalized Scharlemann cycle in D.*

Proof. There is a possibility that ∂D is not a circle. That is, ∂D may contain a double edge, and also more than two edges of ∂D may be incident to a vertex on ∂D. Since we will find a generalized Scharlemann cycle within D, we can cut formally the graph $G_{TK} \cap D$ along double edges of ∂D and at vertices to which more than two edges of ∂D are incident so that ∂D is deformed into a circle. (See also [14, Section 5].) Thus we may assume that ∂D is a circle.

If D is a bigon, then the conclusion is obvious. Therefore, we assume that D has at least three sides.

Suppose that D has a diagonal edge e, which has two distinct labels a and b at its endpoints as in Figure 3(a). Since D is an x-face, $a \neq x$ and $b \neq x$. Without loss of generality, we may assume that the labels appear in counterclockwise order around the boundary of each vertex, and that $a > b$.

Formally, construct a new x-face D' as follows. The edge e divides D into two disks D_1 and D_2. We can assume that D_1 lies on the right side of e when e is oriented from the endpoint with label a. If three labels b, a, x appear in this order around the corners of ∂D, then discard D_2, and insert additional edges to the left of e, and parallel to e, until we first reach label x at one or both ends of an additional edge. See Figure 3(b). The new x-face D' is the union of D_1 and some additional bigons. If three label b, x, a appear in this order, then discard D_1, and insert additional edges to the right of e as above. Then D' is the union of D_2 and some additional bigons.



(a) (b)

FIGURE 4. Γ

Remark that there is no generalized Scharlemann cycle among additional edges and e.

Repeat the above process for every diagonal edge which is not level, then get a new x-face E and a graph Γ in E. All diagonal edges of Γ are level, and all boundary edges are x-edges, especially label x can appear on both ends of a boundary edge. Such boundary edges are called level x-edges to distinguish them from level 0-edges. See Figure 4.

From now on, we assume that there is no Scharlemann cycle in Γ.

Claim 4.2. Γ *contains a level* 0*-edge.*

Proof. Suppose not. Then Γ contains a Scharlemann cycle by [14, Lemma 5.2]. $\square$

Let D_0 be the disk face of $\Gamma - \{\text{level 0-edges} \subset \text{Int}E\}$, which is not a bigon. We will show that for any edge of Γ on ∂D_0, it has labels only on $\{-1, 0, 1\}$ at both ends.

There are consecutive vertices $v_0, v_1, \ldots, v_n$ on ∂D_0 such that v_i is not a base of a level 0-edge for $1 \le i \le n-1$, but v_0 and v_n are base of level 0-edges. Possibly, $n = 1$. See Figure 4(b). Let e_i be the x-edge connecting v_i with v_{i+1} for $i = 0, 1, \ldots, n-1$. Let $r(e_i)$ ($l(e_i)$, resp.) be the label of the end of e_i at v_i (v_{i+1}, resp.) and F_i the family of mutually parallel edges containing e_i. The number of edges in F_i is denoted by $|F_i|$. Also, let f_i be the edge of F_i lying on ∂D_0. We define $l(f_i)$ and $r(f_i)$ similarly as above.

Claim 4.3. (1) *If p is even, then $|F_i| \le p/2$. Furthermore, if $|F_i| = p/2$, then either the x-edge e_i is level, or its label on the other end is $x-1$ or $x+1$.*
(2) *If p is odd, then $|F_i| \le (p+1)/2$. Furthermore, if $|F_i| = (p+1)/2$, then e_i is a level x-edge.*

Proof. These follow from Lemma 2.8 and the definition of F_i. $\square$

Claim 4.4. *If $x = \pm 1$, or $|F_j| = 1$ for some j, then each f_i has labels on $\{-1, 0, 1\}$.*

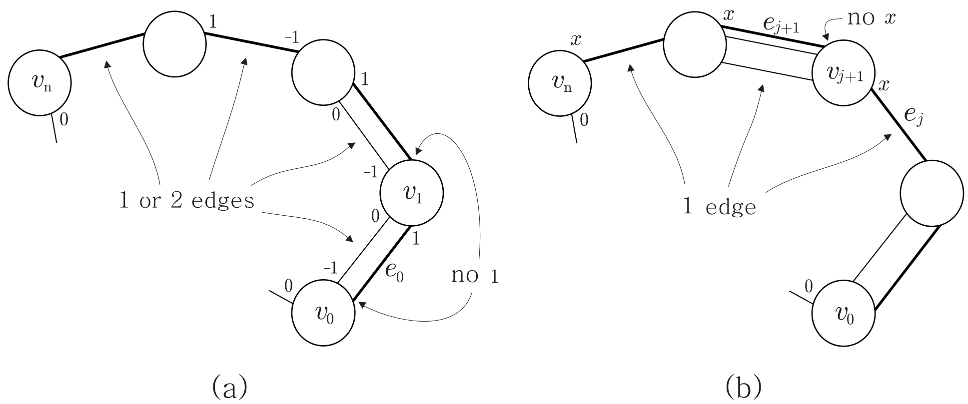


FIGURE 5.

Proof. Suppose $x = 1$. Then $r(e_0) \neq 1$, since F_0 cannot have $p-1$ edges, except the case $p = 3$, by Claim 4.3. Thus $l(e_0) = 1$ and $|F_0| \leq 2$, since there cannot be a level edge or an S-cycle. When $p = 3$, we have the same conclusion. For, $|F_0| = 1$ or 2, and if two then $l(e_0) = r(e_0) = 1$ by Claim 4.3.

The same argument runs until we got that $l(e_i) = 1$ and $|F_i| \leq 2$ for all i. Then it is easy to see that f_i has labels on $\{-1, 0, 1\}$ at both ends (see Figure 5(a)). In particular, when $p = 3$, the corner on ∂D_0 at v_i contains label 0. Similarly for the case where $x = -1$.

Suppose that $|F_j| = 1$ as in Figure 5(b). Assume $l(e_j) = x$. By Claim 4.3, $r(e_{j+1}) \neq x$, and so $l(e_{j+1}) = x$. If $|F_{j+1}| > 1$, then it contains either a level edge or an S-cycle. Thus $|F_{j+1}| = 1$. Hence $|F_{n-1}| = 1$ and $l(e_{n-1}) = x$. This implies that $x = 1$. If $r(e_j) = x$, then a similar argument shows that $|F_0| = 1$ and $r(e_0) = x$, which implies $x = -1$. □

When $p = 3$, then $x = \pm 1$. Then we get the desired result by Claim 4.4. Therefore, we assume $p > 3$. Furthermore, we can assume that $x \neq \pm 1$ and any $|F_i| > 1$ by Claim 4.4.

Claim 4.5. *If $r(e_i) = x$ for all i, then each f_i has labels on $\{-1, 0, 1\}$.*

Proof. Note that $r(f_0) = -1$ and $l(f_{n-1}) = 1$. Consider F_0. Since F_0 contains neither level edge nor S-cycle, $l(f_0) \geq 0$. This implies $|F_0| \leq |F_1|$. Then $l(f_1) \geq l(f_0)$. Thus we have $0 \leq l(f_0) \leq l(f_1) \leq \cdots \leq l(f_{n-1}) = 1$. Hence $l(f_i) = 0$ or 1 for all i. The result immediately follows from this observation. □

Thus we can assume that $r(e_i) = x, 0 \leq i \leq m-1$ and $r(e_m) \neq x$ for some m. (Possibly, $m = 0$.) Then $l(e_m) = x$. See Figure 6. Also, $|F_m| \leq p/2$ by Claim 4.3. Then $r(e_{m+1}) \neq x$. Thus we have $r(e_i) \neq x$ and $l(e_i) = x$ for $m \leq i \leq n-1$, and $|F_{n-1}| \leq p/2$.

Case 1: *p is even.*

Then $|F_0| = |F_{n-1}| = p/2$, since $|F_0| + |F_{n-1}| = p$. See Figure 6(a). Thus $l(e_0) = x - 1$ or x by Claim 4.3. For, if $l(e_0) = x + 1$ then $F(e_0)$ contains an S-cycle.

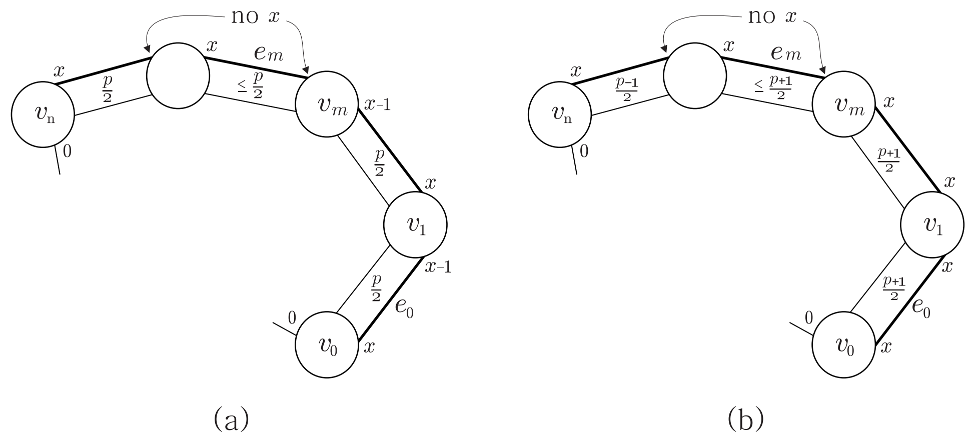


Figure 6.

Indeed, $l(e_0) = x - 1$, since $r(e_1) = x$. Hence $|F_1| = p/2$, and $l(e_1) = x - 1$ or x again. If $r(e_2) = x$, then $l(e_1) = x - 1$ as above. Thus we can conclude that $|F_i| = p/2$ $(0 \leq i \leq m-1)$, and that $l(e_i) = x - 1$, and $l(f_i) = 0$, $r(f_i) = -1$ for $0 \leq i \leq m-2$. Also $l(e_{m-1}) = x$ or $x-1$, and hence $l(f_{m-1}) = 0$ or 1, and $r(f_{m-1}) = -1$. Since $|F_{n-1}| = p/2$ and $l(f_{n-1}) = 1$, we see $r(e_{n-1}) = x+1$ and so $r(f_{n-1}) = 0$. Thus $l(f_i) = 1$ for $m \leq i \leq n-1$, and $r(f_i) = 0$ for $m+1 \leq i \leq n-1$ and $r(f_m) = -1$ or 0.

Case 2: *p is odd.*

Then $|F_0| = (p+1)/2$, and $|F_{n-1}| = (p-1)/2$ similarly for the case where p is even. See Figure 6(b). Also, $l(e_0) = x$ by Claim 4.3(2). Then we see that $|F_i| = (p+1)/2$ and $l(e_i) = x$ for $0 \leq i \leq m-1$. Thus $r(f_i) = -1$ and $l(f_i) = 0$ for $0 \leq i \leq m-1$. Since $|F_{n-1}| = (p-1)/2$ and $l(f_{n-1}) = 1$, we have $r(e_{n-1}) = x+1$ and $r(f_{n-1}) = -1$. Then $l(f_{n-2}) = 0$ and therefore $|F_{n-2}| = (p+1)/2$. By Claim 4.3(2), the only possibility is $m = n-1$.

Thus we have shown that ∂D_0 gives a generalized Scharlemann cycle. This completes the proof of Theorem 4.1. □

Theorem 4.6. *Let $p \geq 3$. Then G_{TK} cannot contain a non-zero x-face.*

Proof. This follows immediately from Lemma 2.7 and Theorem 4.1. □

5. The case $G_{TK} = G_K$

Lemma 5.1. (1) *G_P cannot contain a Scharlemann cycle.*

(2) *G_P cannot contain a generalized S-cycle.*

(3) *G_P has at most two level edges with different labels.*

(4) *Suppose $k \geq 3$. Then G_P has at most $k/2+1$ ($(k+1)/2$, resp.) mutually parallel positive edges if k is even (odd, resp.). Furthermore, if G_P has a family of $k/2+1$ ($(k+1)/2$, resp.) mutually parallel positive edges, then the first and last edge (the first or last edge, resp.) of the family are level, when k is even (odd, resp.).*

Proof. If G_P contains a Scharlemann cycle, then we can get a new Klein bottle in $M(\gamma)$ which meets V_γ fewer than $\widehat{K}$ ([11, Theorem 6.4]). Theorem 6.6 in [11] implies (2). Here, we need the fact that the distance between two Dehn fillings creating projective planes is at most one [18]. (3) is a restatement of Lemma 2.6(2). For (4), if not, such a family contains a generalized S-cycle. □

Lemma 5.2. *If $k \geq 2$, then $p \geq 3$.*

Proof. Assume for contradiction that $p = 2$ because of Lemma 2.1. Let u and v be the vertices of G_P, where u can be a base of negative loops. Since u and v have the same degree Δk, if u has a loop, then so does v. Then there would be a trivial loop. Thus we can see that G_P has no loops. Then G_P consists of at most two families of mutually parallel edges; one is a family of positive edges, and the other is that of negative edges.

If $k \geq 3$, then G_P contains more than k mutually parallel negative edges by Lemma 5.1(4). Then an easy Euler characteristic calculation shows that G_K contains a 0-face and no level edges. As in Case 2 of the proof of Proposition 3.1, this gives a contradiction.

If $k = 2$, then G_P has at most two positive edges. Otherwise, there would be two edges which are parallel in both G_P and G_K. But this implies that M is cabled by [7, Lemma 2.1]. Thus G_P contains at least four negative edges. Similarly, G_K contains a 0-face and no level edges, a contradiction. □

Theorem 5.3. *If $k \geq 3$, then G_P cannot contain a generalized web.*

Proof. Assume for contradiction that G_P contains a generalized web Λ_P, possibly with an exceptional vertex y among boundary vertices of Λ_P. Let D denote a disk support of Λ_P. Lemma 5.1(3) guarantees the existence of a label x such that G_P contains no positive level x-edges.

Consider Λ_P^x consisting of all vertices and x-edges of Λ_P. Since every boundary vertex of Λ_P, except y, has degree at least $(\Delta - 1)k$, it has at least two edges attached with label x. We remark that Λ_P^x may be disconnected. Choose an innermost component G of Λ_P^x (in D), and let H be its block with at most one cut vertex of G.

Let v, e and f be the numbers of vertices, edges, and disk faces of H, respectively. (We view H as the graph in a disk.) Also let v_i, v_∂ and v_c be the numbers of interior vertices, boundary vertices of H and a cut vertex of G in H, respectively. Hence $v = v_i + v_\partial$ and $v_c = 0$ or 1.

Since H has neither a level x-edge nor a generalized S-cycle, each face of H is a disk with at least 3 sides. Thus we have $3f + v_\partial \leq 2e$. Combined with $1 = v - e + f$ because it has only disk faces, we get $e \leq 3v_i + 2v_\partial - 3$. On the other hand we have $2(v_\partial - v_c) + 3v_i \leq e$ because each boundary vertex of H, except a cut vertex of G, has at least two edges attached with label x. These two inequalities give us that $3 \leq 2v_c$, a contradiction. □

6. The case $k = 2$

Consider the case where $k = 2$ and $\Delta \geq 3$ when $G_{TK} = G_K$. Recall that G_P contains a generalized web Λ_P by Proposition 3.1, Theorem 4.6 and Lemma 5.2. We remark that at most two positive edges can be parallel in G_P. For, if there are three mutually parallel positive edges, then G_P contains either an S-cycle or a pair of positive level edges with the same label. The former is impossible by Lemma 5.1. In the latter case, such two edges are also parallel in G_K. But this implies that M is cabled [7, Lemma 2.1]. Also, if there are two parallel positive edges, then both edges must be level.

A positive level edge with label 1 (or 2) in G_P is called a 1-*edge* (2-*edge*, resp.) and a positive non-level edge is called a *mixed edge*.

Lemma 6.1. *Λ_P contains a pair of parallel edges. In particular, these edges are a* 1*-edge and a* 2*-edge.*

Proof. Any interior vertex of Λ_P has degree at least six, since $\Delta k \geq 6$. Also, any boundary vertex of Λ_P, except an exceptional one, has degree at least four. Therefore Λ_P contains a pair of parallel edges by [22, Lemmas 2.3, 3.2]. Then these edges are both level as above. □

Lemma 6.2. *Let x be a vertex of Λ_P.*

(1) *There are no three i-edges at x for each $i = 1, 2$.*
(2) *There are no three* (*positive*) *level edges connecting x to mutually distinct vertices, not x.*

Proof. (1) Suppose that three positive 1-edges, say, are incident to x. Since all negative loops based at vertex 1 in G_K are mutually parallel, there are at least $p+1$ negative loops at vertex 1. This is impossible by Lemma 2.9.

(2) Let v_1 and v_2 be the vertices of G_K. By Lemma 6.1, Λ_P contains a pair of parallel (positive) edges e_1, e_2. Here, e_i is an i-edge for $i = 1, 2$, and e_i connects a vertex a to another b (possibly, $a = b$). Then e_i gives a negative loop based at v_i with the label pair $\{a, b\}$. Remark that all negative loops at v_i are mutually parallel. If Λ_P has another i-edge e connecting x to y, then e_i and e are parallel, and so $y = x \pm (b - a) \pmod p$. This means that any i-edge at x goes to either vertex $x + b - a$ or $x - b + a$. □

Let n_i denote the number of negative loops at vertex i in G_K. Without loss of generality, we can assume that $n_1 \geq n_2$.

Lemma 6.3. *Let x be a vertex of Λ_P. If a level edge e and a pair of parallel edges are incident to x successively in Λ_P, then e is a* 1*-edge. Furthermore, there are no consecutive pairs of parallel edges at x in Λ_P.*

Proof. Let e_1, e_2 be these pair edges, where e_i is an i-edge. Assume for contradiction that e is the consecutive 2-edge next to e_1. As in the proof of 6.2(2), we may assume that e connects x to $x + r$ and e_1 and e_2 connect x to $x - r$ for some r. Let α, β, α' and β' be the endpoints of these edges. Let $\overrightarrow{\alpha\beta}$ be a properly oriented arc from α to β along ∂P, and $\overrightarrow{\alpha'\beta'}$ similarly. See Figure 7(b).

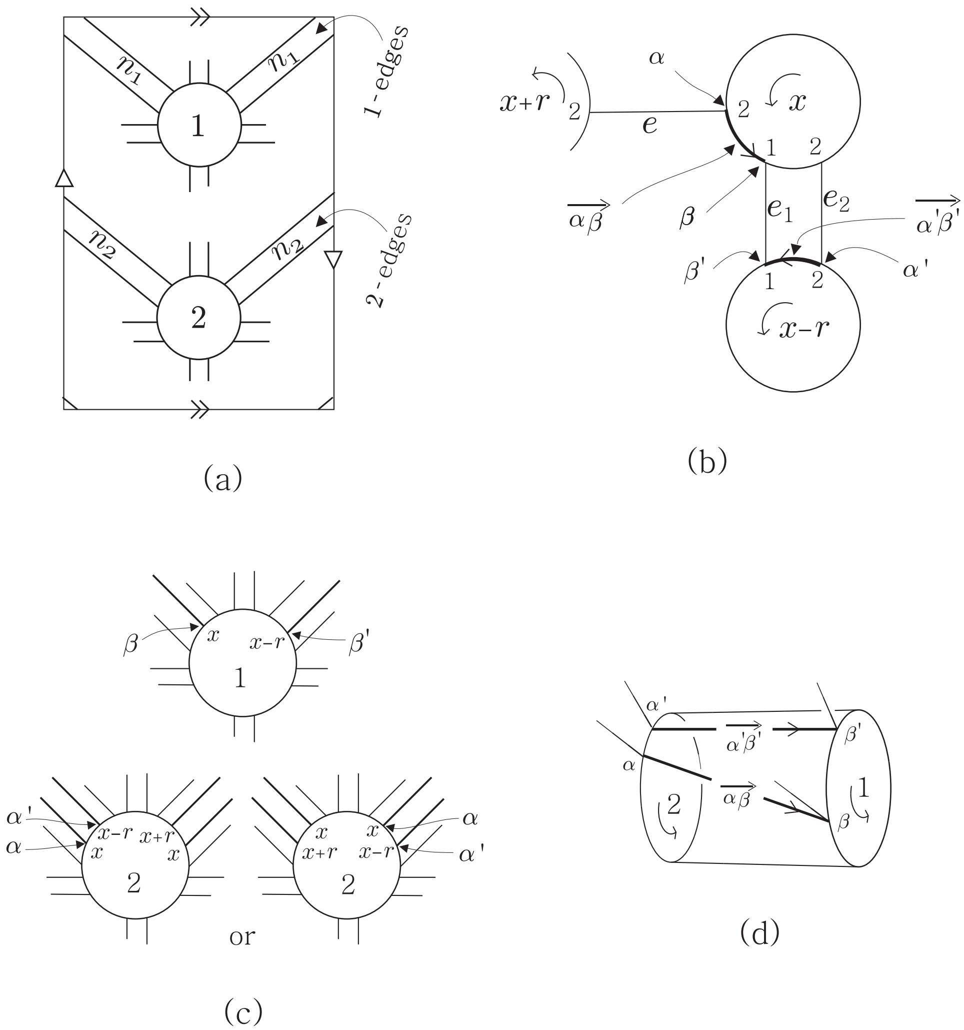


FIGURE 7.

In G_K, e_1 is a negative loop at vertex 1, and its endpoint β has label x, and β' has $x-r$. Also, e and e_2 are contained in the family of mutually parallel negative loops at vertex 2 in G_K, and α and α' appear at the same end of this family. Otherwise, the family contains more than p edges, contradicting Lemma 2.9.

Let $\partial_i K$ be the boundary component of K with label i ($i = 1, 2$). We may assume that $\partial_2 K$ is oriented counterclockwise as in Figure 7(c). Then the subarc $\alpha'\alpha$, with the induced orientation from $\partial_2 K$, contains at most n_2 edge endpoints.

Consider the annulus part of $\partial_0 M$ between $\partial_1 K$ and $\partial_2 K$, containing $\overrightarrow{\alpha\beta}$ and $\overrightarrow{\alpha'\beta'}$. Then we see that the two oriented subarcs $\alpha'\alpha$ and $\beta'\beta$ contain the same number of points in $\partial P \cap \partial K$. See Figure 7(d). But, the subarc $\beta'\beta$ contains at least $n_1 + 1$ edge endpoints.

The second conclusion immediately follows from the first one. $\square$

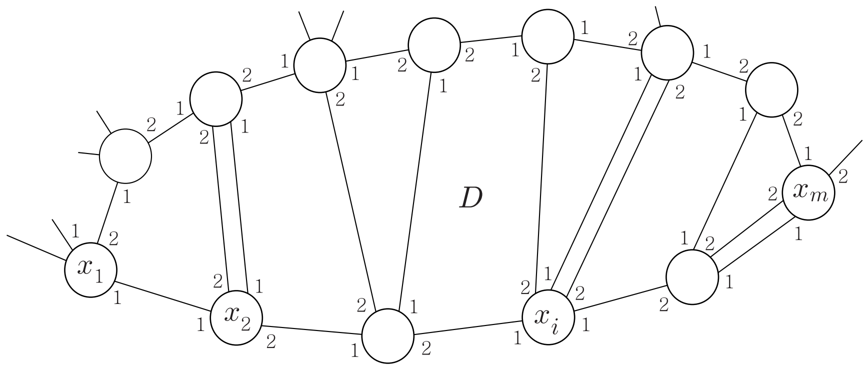


FIGURE 8.

Lemma 6.4. Λ_P *has the following properties.*

(1) *Every disk face is bounded by mixed edges and a non-zero even number of level edges.*
(2) *There is no disk bounded by a cycle consisting of mixed edges, except at most one edge, and containing no vertex in its interior.*

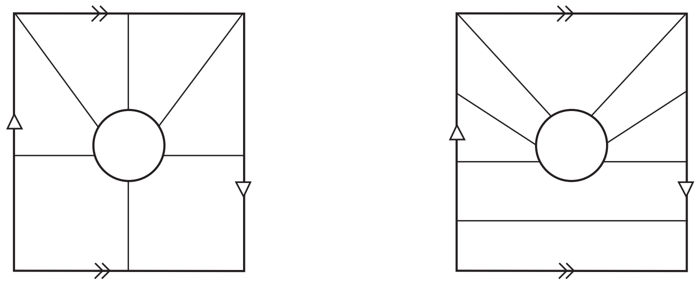
Proof. (1) If a disk face is bounded by only mixed edges, then the boundary cycle is a Scharlemann cycle, which is impossible by Lemma 5.1(1). If the boundary cycle contains an odd number of level edges, we cannot put the labels correctly.

(2) Suppose that there is such a disk D. Since its interior contains no vertex of Λ_P, there is an outermost disk face of Λ_P whose boundary edges consist of mixed edges and at most one diagonal edge (or exceptional one), contradicting (1). □

Suppose that Λ_P has no interior vertex. Then, Λ_P has a boundary vertex x of degree two in $\overline{\Lambda}_P$ by [24, Lemma 3.2]. Then there are two consecutive pairs of parallel edges incident to x in Λ_P. This contradicts Lemma 6.3. Thus Λ_P has an interior vertex. Lemma 6.2(1) implies that each interior vertex has at least two mixed edges.

Consider a maximal path consisting of mixed edges starting at an interior vertex. By Lemma 6.2(2), both ends will reach some boundary vertices. Thus we have a consecutive sequence of mixed edges between two boundary vertices. Then we can choose an outermost disk D so that its interior contains no vertex and no mixed edges, and its boundary consists of a consecutive sequence of mixed edges and a consecutive sequence of boundary edges. Furthermore, we can assume that D does not contain an exceptional vertex of Λ_P.

In D, every diagonal edge of $\overline{\Lambda}_P$ connects an interior vertex and a boundary vertex of Λ_P. For, if there is a diagonal edge connecting two interior vertices, then it contradicts Lemma 6.4(2). Also, if there is a diagonal edge connecting two boundary vertices, then between these two boundary vertices there must be a vertex which has degree two in $\overline{\Lambda}_P$, contradicting Lemma 6.3. Recall that boundary vertices has degree at least four in Λ_P. See Figure 8.



(a) (b)

FIGURE 9. $\overline{G}_K$

Let $x_1, x_2, \cdots, x_m$ be the boundary vertices from left to right. There are at least three boundary vertices by Lemma 6.4(2) and each vertex $x_i, i = 2, \cdots, m-1$, has diagonal edges (so level edges) incident to by Lemma 6.3. Thus all boundary edges between x_2 and x_{m-1} are mixed edges, and the other boundary edges are level edges by Lemma 6.4(1). Furthermore vertices x_2 and x_{m-1} have a pair of parallel edges and a consecutive level edge e. By Lemma 6.3, e is a 1-edge. But there must be another vertex x_i between x_2 and x_{m-1} which has three consecutive level edges, so that the level edge not of the pair is a 2-edge. This contradicts Lemma 6.3.

Theorem 6.5. *If $k \geq 1$, then $\Delta \leq 2$.*

Proof. By Lemma 5.2, $p \geq 3$. If $k \geq 3$, then Proposition 3.1, Theorems 4.6 and 5.3 give a contradiction. We have just shown that the case $k = 2$ is impossible. □

7. THE CASE $k = 1$

Theorem 7.1. *If $k = 1$, then $\Delta \leq 3$.*

Proof. Assume that $\Delta \geq 4$. The reduced graph $\overline{G}_K$ is a subgraph of the graphs shown in Figure 9.

If p is even, there are at most $p/2$ positive edges by Lemma 2.8, and at most $p-1$ mutually parallel negative edges by Lemma 2.9. Thus $\Delta p \leq p + 4(p-1) = 5p - 4$, and then $\Delta = 4$ and $p \geq 4$. Similarly, if p is odd, we have that $\Delta = 4$ and $p \geq 3$. It follows that G_K contains at least two positive edges.

First, assume that $\overline{G}_K$ is a subgraph of Figure 9(a). This family of mutually parallel positive edges in G_K contains a generalized S-cycle, contradicting Lemma 2.7. Next, assume that $\overline{G}_K$ is a subgraph of Figure 9(b). Since each family of mutually parallel negative edges contains at most $p-1$ edges, we can assume that the labels are as in Figure 10. Then the family of mutually parallel positive edges contain an S-cycle with label pair $\{p-2, -1\}$. □

Thus we have proved Theorem 1.2.

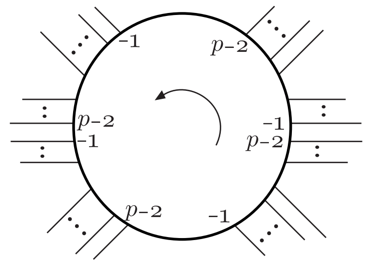


Figure 10. Labels in G_K

8. The case $G_{TK} = G_T$

An *extended S-cycle* in G_P is the quadruple $\{e_1, e_2, e_3, e_4\}$ of mutually parallel positive edges in succession and $\{e_2, e_3\}$ forms an S-cycle.

Lemma 8.1. *Let $t \geq 3$.*

(1) *G_P cannot contain an extended S-cycle.*
(2) *G_P has at most 3 labels which are labels of S-cycles.*
(3) *G_P has at most $t/2 + 1$ mutually parallel positive edges.*

Proof. (1) is shown in [1, Lemma 2.10].

(2) If there are four such labels, there are two S-cycles with disjoint label pairs. Then $M(\beta)$ contains a Klein bottle as in the proof of [11, Lemma 3.10]. (Recall that $M(\beta)$ is irreducible.) By Theorem 1.2, $M(\beta)$ must contain a Klein bottle F which meets the core k_β of the attached solid torus V_β in a single point, since $\Delta \geq 3$. Then the torus $\partial N(F)$ is essential in $M(\beta)$. Otherwise $M(\beta)$ would be either a lens space containing a Klein bottle or a prism manifold, both of which are not toroidal. This essential torus meets k_β in two points. This contradicts the minimality of t.

(3) follows from (1) and (2), since a family of $t/2 + 2$ mutually parallel positive edges contains either an extended S-cycle or two S-cycles with disjoint label pairs. □

Lemma 8.2. *If $t \geq 3$, then $p \geq 3$.*

Proof. This follows from the same argument as in the proof of Lemma 5.2. We use Lemma 8.1(3) instead of Lemma 5.1(4). □

Theorem 8.3. *If $t \geq 3$, then G_P cannot contain a generalized web.*

Proof. Assume for contradiction that G_P contains a generalized web Λ_P, possibly with an exceptional vertex y of G_P^+. Let D denote a disk support of Λ_P.

Let x be a label of G_P. Suppose that G_P has no S-cycle with a label x. Consider Λ_P^x, consisting of all vertices and x-edges of Λ_P as in the proof of Theorem 5.3. Choose an innermost component G of Λ_P^x (in D), and let H be its block with at most one cut vertex of G. Since H has neither S-cycle with label x nor extended S-cycle, each face of H is a disk with at least 3 sides. Then the same calculation as

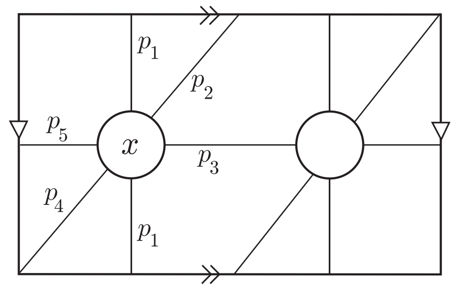


FIGURE 11. $\overline{G}_T$

in the proof of Theorem 5.3 gives a contradiction. Therefore G_P has t labels which are labels of S-cycles. Note that the existence of an S-cycle in G_P guarantees that t is even [1, Lemma 2.2] and so $t \geq 4$. This contradicts Lemma 8.1 (2). □

Theorem 8.4. *If $t \geq 3$, then $\Delta \leq 2$.*

Proof. By Lemma 8.2, $p \geq 3$. Then Proposition 3.1, Theorems 4.6 and 8.3 give a contradiction. □

9. THE CASE $t = 2$

Theorem 9.1. *If $t = 2$, then $\Delta \leq 3$.*

Proof. The reduced graph $\overline{G}_T$ is a subgraph of the graph shown in Figure 11 [7, Lemma 5.2].

Here, $p_i \geq 0$ denotes the number of edges in each of the families of mutually parallel edges. Then $\Delta p = 2p_1 + p_2 + p_3 + p_4 + p_5$. We have that $p_1 \leq (p+1)/2$ and $p_i \leq p-1$ for $i = 2, 3, 4, 5$ by Lemmas 2.8 and 2.9. Thus, $\Delta p \leq (p+1) + 4(p-1) = 5p - 3$. This inequality implies that $\Delta = 4$ and all p_i's are non-zero.

Claim 9.2. *All non-loop edges of G_T are negative.*

Proof. Assume not. Let x and y be the vertices of G_T. If p is even, then $4p = \Delta p \leq 6p/2 = 3p$, which is absurd. If p is odd, then $4p = \Delta p \leq 6(p+1)/2 = 3p + 3$. Thus $p = 3$. But then, each of six families at x, say, consists of two edges, and we have an S-cycle in the family of loops at vertex x. □

Without loss of generality, we can assume that $p_1 + p_2 + p_3 \geq 2p$. Let $r \equiv p_1 + p_2 + p_3 + 1 \pmod p$. Since $2p < p_1+p_2+p_3+1 \leq (p+1)/2+2(p-1)+1 < 3p$, we see $p_1+p_2+p_3+1 = r+2p$. Then $p_1 = r+2p-(p_2+p_3+1) \geq r+2p-2(p-1)-1 = r+1$. Hence $1 \leq r \leq p_1 - 1$. Thus the loop family around x, say, contains a generalized S-cycle. □

10. THE CASE $t = 1$

Theorem 10.1. *If $t = 1$, then $\Delta \leq 1$.*

Proof. Assume that $\Delta \geq 2$. Since $t = 1$, G_T can have only positive edges, and so G_P has only negative edges. Note that G_T has at most three families of mutually parallel edges by [8, Lemma 5.1]. Let A, B and C be such three families, and $|A|$, $|B|$ and $|C|$ be the number of edges of each family. Without loss of generality, we can assume that A has at least one edge.

First assume that Δ is even. If $|A| > 1$, then A contains a generalized S-cycle by the symmetry of labels around the vertex. Thus $|A| = 1$. Similarly we have $|B| \leq 1$, $|C| \leq 1$, and therefore $\Delta p \leq 6$. Since Δ is even and $p \geq 2$, we have $\Delta = 2$ and $p = 2, 3$. Then G_T has two or three level edges with different labels, contradicting Lemma 2.6(1).

Assume now that Δ is odd. Since $\Delta p/2$ is integral, p must be even. Thus $|A| \leq p/2$ by Lemma 2.8, and similarly for B, C. Then $\Delta p/2 = |A| + |B| + |C| \leq 3p/2$. Hence we have $\Delta = 3$ and $|A| = |B| = |C| = p/2$. This implies that G_T contains two Scharlemann cycles of length three, which is a contradiction. □

We have thus completed the proof of Theorem 1.1.

11. Examples

In this section we give examples showing that our estimates are sharp.

Example 11.1. [5, Theorem 4.2] shows that there is a hyperbolic manifold M, admitting two Dehn fillings $M(\alpha)$ and $M(\beta)$ such that $M(\alpha) = L(3,1)\sharp L(2,1)$ and $M(\beta)$ is toroidal, and $\Delta(\alpha, \beta) = 3$. In fact, $M(\beta)$ contains an essential torus which bounds $Q(2,-4)$ and $Q(2,2)$, where $Q(r,s)$ is a Seifert fibered manifold with orbifold a disk with two cone points of index r and s. Also, $Q(2,2)$ contains a Klein bottle, and so does $M(\beta)$. It is not hard to see that $M(\beta)$ contains an incompressible torus hitting the attached solid torus V_β twice, and a Klein bottle hitting V_β once from the construction using tangles.

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